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THE RESTRICTIONS OF THE PARAMETERS OF CAVITATIONAL FLOW*

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The stability principle is used to study the admissible values of the physical parameters defining cavitational flow. An example is given of a non-stationary flow for which cavitational flow is possible only up to a certain instant. The restrictions on the cavitational flow parameters are studied in the light of surface-tension effects. For different Weber numbers we calculate the size of the cavitational cavity behind a cone in the flow of weightless fluid for zero cavitation number.

The symmetric inviscid flow past a body of ideal incompressible heavy fluid with jet separation is reduced to the following boundary value problem for the disturbed flow potential:

$$\Delta\varphi = 0;\quad \frac{\partial\varphi}{\partial r}\Big|_{\Sigma_1+\Sigma_2} = \frac{\partial R}{\partial x}\left[\frac{\partial\varphi}{\partial x}\Big|_{\Sigma_1+\Sigma_2} + V_\infty(t)\right] + \frac{\partial R}{\partial t} \tag{0.1}$$
$$\left[\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\operatorname{grad}\varphi)^2\right]\Big|_{\Sigma_2} = \frac{\sigma(t)}{2} \pm \frac{x}{\mathrm{Fr}^2}$$

Here, Σ_1 is the surface of the cavitator, Σ_2 in the free surface, σ, Fr are the cavitation and Froude numbers, $R(x, t)$ is the width or radius of the cavern for the plane or axisymmetric case respectively; the axis of symmetry x coincides with the velocity direciton at infinity $V_\infty(t)$.

It is very difficult to study the non-linear boundary value problem (0.1) with a non-stationary free surface. In particular, there is as yet not exhaustive study of the values of the physical parameters σ, Fr, V_∞, defining the flow, for which a solution exists and is unique (in the non-stationary case the cavitation number and V_∞ are functions of time, and the unique solvability of (0.1) poses even greater difficulties).

Results concerning plane steady flow are generalized in /1-3/. For this case, the apparatus of the theory of functions of a complex variable is used; as a rule, the solution is linked with the constants, chosen in the plane of the parametric variable. It is difficult to find the connection between these constants and the physical parameters (numbers σ, Fr, and the cavitator geometry) because a system of non-linear equations has to be solved.

Though a lot of attention has been paid to symmetric plane steady cavitation flows of a weightless fluid with fixed points of jet convergence, some questions are still unanswered. For instance, in the case of a wedge of unrealizable flow with $\sigma \leqslant 0$, or for a body with a negative derivative with respect to the coordinate of the cavitator thickness at the jet convergence point R_4, we do not know why negative cavitation numbers of small modulus are observed experimentally, whereas the results of /3/ reveal the existence of a solution for a body of any shape with any cavitation number $\sigma > -1$.

By linearizing problem (0.1) in the case of a thin cavitator or cavern, we can greatly simplify the solution and the study of problems of existence. In particular, it was shown in /4-6/ that steady plane flow past a wedge of ascending heavy fluid is only possible for cavitation numbers which satisfy the condition $\sigma \geqslant \sigma_m(\mathrm{Fr}) > 0$, the minimum cavitation number σ_m being a function of the Froude number and the cavitator shape. The limiting shapes of caverns behind point nozzles were studied in /7/. The minimum cavitation numbers were calculated in /8/ for axisymmetric nozzles, and it was shown that, behind bodies with a negative derivative of the radius, at the jet convergence point $(R_4 < 0)$, there form, even in weightless fluid, stationary caverns with negative σ, though here $\sigma \geqslant \sigma_m$ (the value of σ_m is negative and depends solely on the cavitator shape). In particular, for $\sigma = 0$ behind a body with $R_4 < 0$, a cavern of finite length is formed. It appears that the same conclusions can be drawn in the plane case.

1. The stability principle and constraints on the cavitation flow parameters.

The study of the admissible parameter values defining cavitational flow can be illustrated by the example of the flow of a weightless fluid past a thin cone. The solution of the equation of the first approximation for the cavern radius $R(x)$ may be written as /8/ (R_4 is the tangent of the semi-angle of the cone)

$$R^2 = \sigma x^2/(2\ln\varepsilon) + 2R_4 x + 1 \tag{1.1}$$

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All the lengths here are referred to the radius of the bottom cut of the cone. The small parameter ε is the characteristic value of the derivative of the radius with respect to the coordinate. In particular, we can put $\varepsilon = R_4$.

For $\sigma > 0$, Eq.(1.1) describes caverns of finite length, since $(\sigma/(2\ln\varepsilon)) < 0$, $R_4 > 0$. The unrealizability of negative cavitation numbers can be explained by starting from the stability principle for problems of mathematical physics, according to which small variations of the defining parameters must lead to small variations of the solution.

In the case $\sigma \leqslant 0$, $R_4 > 0$, this principle is violated for the solution of (1.1). For, in this case (1.1) describes caverns of infinite length, and by choosing sufficiently large x, we can arrange for the difference in the solutions to be as large as desired, even if the σ_1 and σ_2 defining them are infinitely close.

This principle can explain the constraints obtained in /8/ on the cavitation number $\sigma \geqslant \sigma_m$ $(\sigma_m < 0)$ for axisymmetric bodies with $R_4 < 0$ and the constraints described in /4, 8/ in the case of heavy fluid flow. It should be said that Kirchhoff's solution /9/ for the steady plane flow of a weightless fluid past a plate with $\sigma = 0$ is also an example of an unstable solution, since, with values of cavitation number infinitely close to zero, the cavern has restricted size, i.e., differs as much as desired from Kirchhoff's infinite cavern.

In short, the stationary solutions for wedges and cones $(R_4 \geqslant 0)$ with infinite caverns $(\sigma = 0)$ are of merely theoretical interest, since the instability in question can lead to the solutions being capable of giving different results at $\sigma = 0$, though they describe the cavern shape fairly well for $\sigma > 0$. This may explain the fact that, with $\sigma = 0$ and as $x \to \infty$, (1.1) gives the asymptotic behaviour

$$R(x) \to \sqrt{2R_4} \cdot \sqrt{x},$$

which is not in accord with the well-known Levinson-Gurevich-Yakimov asymptotic behaviour /10-12/.

2. Restrictions in the non-stationary case. A similar principle of stability can also be used in the non-stationary case: if, in a process with continuously varying functions $\sigma(t)$, $V_\infty(t)$ an instant t_c occurs when the cavern changes its shape with a "jump" (e.g., from finite to infinite), then the flow can be longer be realized when $t \geqslant t_c$. Since the cavern shape in the non-stationary case depends, not only on the running values of the parameters, but also on the prehistory of the flow, we can speak of restrictions on the parameters in this case only in the context of a specific process.

As an example we consider the axisymmetric flow past a thin body of ascending fluid with constant velocity at infinity and linearly decreasing cavitation number $\sigma(t) = \sigma(0) - a_0 t$. If we use the expressions in /13/, the solution of the non-stationary equation of the first approximation takes the form

$$R^2(x,t) = \frac{\sigma(t)\,x^2}{2\ln\varepsilon} + \frac{x^3(a_0 - \mathrm{Fr}^{-2})}{3\ln\varepsilon} + 2R_4x + 1 \tag{2.1}$$

Simple analysis shows that, when the parameter a_0 satisfies the condition $0 < a_0 < \mathrm{Fr}^{-2}$, an instant t_c occurs such that, for $t > t_c$, the cavern described by Eq.(2.1) becomes infinite, though before this instant the size of the cavity is bounded. We can calculate the critical value of the cavitation number $\sigma_c = \sigma(t_c)$. For instance, in the case when $R_4 = 0$,

$$\sigma_c = [-6(\mathrm{Fr}^{-2} - a_0)^2 \ln\varepsilon]^{1/3}$$

In this case, therefore, a reduction of the cavitation number to a value less than σ_c is impossible. With other combinations of the parameters Fr and a_0, it is not possible to indicate processes in which cavitational flow occurs only up to a certain instant, and then becomes impossible. For instance, in the case of a weightless fluid, it follows from Eq.(2.1) that, as the cavitation number falls $(a_0 > 0)$ the caverns will have finite size with both positive and negative running values of the cavitation number. As the cavitation number increases, the cavern size falls continuously.

It follows in particular from what has been said that non-stationary flow of weightless fluid past a cone is possible for negative cavitation numbers, though in the stationary case, for bodies with $R_4 \geqslant 0$, the condition $\sigma > 0$ must hold.

Our example does not exhaust the cases of axisymmetric flow for which non-stationary cavitational flow is only possible up to a certain instant. For a general study of this topic, the expressions of /13/ for the general solution of the equation of the first approximation may be used. The critical values of the parameters of axisymmetric cavitational flows may be refined by using the equation of the second approximation /14/.

3. The influence of surface tension. Some results concerning the unique solvability of the plane problem of steady cavitational flow, in the context of capillary effects, may be found in /3/. Here we take the axisymmetric case with $R_4 > 0$. The shape of a thin axisymmetric stationary cavern when surface tension is taken into account is described by the equation /15/

$$[RR'' + R'^2]\ln\varepsilon + \frac{\mathrm{We}}{2}\left\{\frac{R''}{(1+R'^2)^{3/2}} - \frac{1}{R(1+R'^2)^{1/2}}\right\} = \frac{\sigma}{2} \pm \frac{x}{\mathrm{Fr}^2} \tag{3.1}$$
$$\mathrm{We} = 2\varkappa'/(\rho' V_\infty'^2 R_3')$$

in which the prime denotes derivatives with respect to the coordinate; all lengths are referred to the radius of the bottom cut of the cavitator R_3'; $\varkappa'$ is the surface tension.

In cases of practical interest, Weber's number We is extremely small, so that we can leave only the leading term $1/R$ in the braces and obtain, for a weightless fluid,

$$\ln\varepsilon\, d^2R^2/dx^2 - \mathrm{We}/R = \sigma \tag{3.2}$$

After reducing the order of Eq.(3.2) and integrating, we arrive at

$$(dR^2/dx)^2 = a - bR - cR^2 \tag{3.3}$$
$$a = 4R_4^2 - 2\frac{\sigma + 2\,\mathrm{We}}{\ln\varepsilon}, \quad b = -\frac{4\,\mathrm{We}}{\ln\varepsilon}, \quad c = -\frac{2\sigma}{\ln\varepsilon}$$

In the centre section of the cavern the left-hand side of (3.3) is zero, so that when $\sigma > 0$ the maximum cavern radius R_m is given by

$$R_m^2 = 1 - 2\sigma^{-1}R_4^2\ln\varepsilon + 2\sigma^{-1}\,\mathrm{We}\,(1 - R_m) \tag{3.4}$$

The presence of non-zero surface tension when $\sigma > 0$ leads to a reduction in R_m, since the first two terms on the right-hand side of (3.4) give a value of the maximum cavern radius $R_m > 1$ when $\mathrm{We} = 0$. This conclusion agrees with the results in /15, 16/.

It follows from (3.3) that the cavern shape is symmetric about the centre section, and after integration the following dependence can be obtained on the piece from the jet convergence point $(x = 0)$ to the centre section:

$$x = c^{-1}\sqrt{a - b - c} - {}^1\!/_2 c^{-3/2}\sqrt{\Delta^2 - (2Rc + b)^2} + \frac{1}{2}bc^{-3/2} \times \tag{3.5}$$
$$\{\arcsin[(2c + b)/\Delta] - \arcsin[(2Rc + b)/\Delta]\}, \quad \Delta = \sqrt{4ac + b^2}$$

When obtaining Eqs.(3.3), (3.5), the standard boundary conditions at the jet convergence point: $R = 1$, $R' = R_4$, have been used.

The stability principle enables us in the case $R_4 > 0$ to analyse the admissible values of the parameters σ and We, defining the flow. It follows from (3.3), (3.5) that the cavern has limited size, continuously dependent on the parameters, in all but the case when polynomial $f(R)$ on the right of (3.3) has no root for $R > 1$ (in this case the cavern becomes infinite).

Analysis of the polynomial $f(R)$ shows that, for $\sigma > 0$, the stability is not destroyed for any value of the Weber number. For $\sigma < 0$ the stability is destroyed if $\mathrm{We} < W_* = -\sigma + \sqrt{2\sigma R_4^2\ln\varepsilon}$, so that cavitational flow is possible in this case only for numbers $\mathrm{We} \geqslant W_*$.

When $\sigma = 0$ the cavern becomes infinite only if $\mathrm{We} = 0$; all other values of the Weber number are admissible. The expression for the cavern radius may then be obtained from (3.3):

$$x = \frac{2}{3b^2}(z^3 - 3a_1 z) - \frac{4R_4}{3b^2}(4R_4^2 - 3a_1) \tag{3.6}$$
$$z = \sqrt{a_1 - bR}, \quad a_1 = 4(R_4^2 - \mathrm{We}/\ln\varepsilon)$$

Computations from (3.6) for a cone with $R_4 = 0.1$ (it was assumed that parameter ε is equal to the ratio of R_m to cavern length L)

We	$5\cdot10^{-2}$	10^{-2}	10^{-3}	10^{-4}	10^{-6}
L	10,0	305	$6,3\cdot10^4$	$1,2\cdot10^7$	$2,8\cdot10^{11}$
R_m	1,4	5,1	69	945	$1,4\cdot10^5$

reveal that, when surface tension is present, caverns of finite size are possible behind the cones even at zero cavitation number.

REFERENCES

1. BIRKHOFF G. and ZARANTONELLO E.H., Jets, Wakes and Cavities, Academic Press, 1957.
2. GUREVICH M.I., Theory of jets of ideal fluid (Teoriya strui ideal'noi zhidkosti), Nauka, Moscow, 1979.
3. MONAKHOV V.N., Boundary value problems with free boundaries for elliptic systems of equations (Kraevye zadachi so svobodnymi granitsami dlya ellipticheskikh sistem uravnenii), Nauka, Novosibirsk, 1977.
4. ACOSTA A.I., The effect of longitudinal gravitational field on the supercavitating flow over a wedge, Trans. ASME, Ser. E, 28, 2, 1961.

5. LENAU C.W. and STREET R.L., A non-linear theory for symmetric, supercavitating flow in a gravity field, J. Fluid Mech., 21, pt.2, 1965.
6. KISELEV O.M. and GUBAIDULLINA R.Z., Cavitational flow past a wedge in any gravitational field, in: Proceedings of a Seminar on Boundary Value Problems, Izd-vo Kazan. un-ta, Razan', 1969.
7. EFREMOV I.I. and SEREBRYAKOV V.V., On the shape of thin caverns at small cavitation numbers in plane and axisymmetric flows, in: Hydromechanics, Nauk. dumka, Kiev, 1978.
8. NESTERUK I.G., On the shape of a thin axisymmetric cavern in heavy fluid, Izv. Akad. Nauk SSSR, MZhG, 6, 1979.
9. KIRCHHOFF G., Zur thoerie freier flussigkeitsstrahlen, J. reine und angew. Math., 70, 1869.
10. LEVINSON N., On the asymptotic shape of the cavity behind an axially symmetric nose moving through an ideal fluid, Ann. Math., 47, 4, 1946.
11. GUREVICH M.I., Flow past an axisymmetric half-body of finite drag, PMM, 11, 1, 1947.
12. YAKIMOV YU.L., On the integral of energy in motion with small cavitation numbers and limiting cavern shapes, Izv. Akad. Nauk SSSR, MZhG, 3, 1983.
13. NESTERUK I.G., On the shape of a thin axisymmetric non-stationary cavern, Izv. Akad. Nauk SSSR, MZhG, 4, 1980.
14. NESTERUK I.G., Determination of the shape of a thin axisymmetric cavern on the basis of an integro-differential equation, Izv. Akad. Nauk SSSR, MZhG, 5, 1985.
15. NESTERUK I.G., Some proglems of axisymmetric cavitational flows, Izv. Akad. Nauk SSSR, MZhG, 1, 1982.
16. BUIVOL V.N. and SHEVCHUK YU.R., Interaction of surface tension and gravity in cavitational flows, in: Hydromechanics, Nauk. dumka Kiev, 1978.

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ASYMPTOTIC SOLUTION OF BOUNDARY VALUE PROBLEMS FOR WEAKLY PERTURBED WAVE EQUATIONS*

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A method that is asymptotic with respect to the small parameter ε is proposed for solving boundary value problems for weakly linear equations with partial derivatives. Linear travelling waves, defined when $x \geqslant 0$, $t \geqslant 0$ and which only interact on the boundary $x = 0$, are the solution of the unperturbed problems. An asymptotic solution which is uniformly suitable for $t, x = O(\varepsilon^{-1})$ is constructed for the perturbed problem using the method of averaging along the characteristics. A model problem of gas dynamics is considered - the problem of the motion of a piston in a semi-infinite tube.

The problems considered in this paper are usually solved by the method of regular expansion with respect to the parameter ε /1, 2/. However this method leads to secular terms appearing in the asymptotic solution, which makes the latter unsuitable for values of the arguments $t, x = O(\varepsilon^{-1})$. But large values of t and x are more interesting when analysing weakly linear waves, since the non-linear, dissipative and other factors, which are usually disregarded in the simplest linear models, begin to develop. Asympotic methods enabling us to solve Cauchy's problem for the equations considered below were proposed in /3-6/. Boundary value problems of the "resonator" type were solved in /7, 8/. The technique proposed previously is modified below for problems in which $x \geqslant 0$.

1. In practice, problems in which two travelling waves weakly interact are the ones most frequently analysed. Suppose the behaviour of these waves is described by problem

$$r_t + r_x = \varepsilon f_1[r, s, \varepsilon], \quad s_t - s_x = \varepsilon f_2[r, s, \varepsilon], \quad t > 0, \quad x > 0 \tag{1.1}$$

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